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DYNAMIC MODELLING OF A NON-LINEARLY CONSTRAINED FLEXIBLE MANIPULATOR WITH A TIP MASS BY HAMILTON'S PRINCIPLE

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In this paper, the equations of motion for a non-linearly constrained flexible manipulator with a tip mass are derived by using Hamilton's principle. Dynamic formulation is based on expressing the kinetic and potential energies of the manipulator system in terms of generalized co-ordinates. Four dynamic models, based on Timoshenko, Euler, simple flexure and rigid body beam theories are used to describe the flexible two-link and single-link manipulators. The Lagrange multiplier method is employed to treat the problem with geometric constraint. The emphasis of this paper is that the generalized friction force is taken into account only whilst the manipulator is in contact with the constrained surface. It is found that the rigid body motion and flexible vibrations are non-linearly coupled in the equations of motion. Some observations are also discussed.

1. INTRODUCTION

Industrial robots and manipulators are used to perform a variety of tasks which include painting, spraying, grinding, etc. These have been traditionally designed on the basic assumption that all members are rigid bodies and the dynamic equations have been derived by many researchers. In particular, industrial robots with lightweight and flexible links are of extreme importance for many industrial applications. There is a current trend toward the development of lightweight robot and manipulator arms. The advantages of a lightweight arm are lower initial and operating costs, greater mobility, and higher operating speeds. The reduction of the component weight allows the actuators to move faster and carry heavier loads with longer links.

The derivation of the dynamic equation of motion for flexible manipulators has been extensively studied by many researchers, and those include the following: Matsuno *et al*. [1] proposed a method for the hybrid position/force control of planar manipulators with two flexible links which are in contact with a constrained surface. Yuan [2] considered the rest-to-rest maneuver of a horizontally slew torque-driven beam undergoing geometrically exact elastic deflections. The equations of motion for robot manipulators consisting of both rigid and flexible links are derived by Low and Vidyasagar [3]. Wang and Guan [4] presented the

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influence of rotating inertia, shear deformation and tip load on the vibration behavior of a one-link flexible manipulator. Wright *et al*. [5] used the method of Frobenius to solve for exact frequencies and mode shapes for a rotating beam in which both the flexural rigidity and the mass distribution vary linearly. Choura *et al*. [6] derived a set of governing differential equations for the in-plane motion of a rotating thin flexible beam. Stockton and Garcia [7] focused on the physically relevant case of a flexible link undergoing periodic slewing motion. Anderson [8] investigated the stability of a manipulator subject to a non-conservative force applied at its free extremity. Benati and Morro [9] provided a systematic, thorough procedure for the derivation of dynamical equations for a chain of flexible links. Park and Asada [10] addressed the integrated structure/control design of two-link robot arms for high speed position. A model of a constrained rigid–flexible robotic manipulator suitable for simulation and controller design was developed by Hu and Ulsoy [11]. Damaren and Sharf [12] presented and classified the inertial and geometric non-linearities that arise in the motion and constraint equations for multibody systems. In these previous studies, most neither considered the manipulator in contact with constrained surface nor investigated the friction force between the end-effector and constrained surface.

In order to investigate the link deformations and the characteristics of distributed parameter systems, establishing the dynamic models for flexible manipulators is very important. The primary contribution of this paper is to provide the general forms of a non-linearly constrained flexible arm with a tip mass. This paper presents a procedure for deriving the four dynamic equations for both single-link and two-link flexible manipulators. These are Timoshenko beam model, Euler beam model, simple flexure model and rigid body model. First, the rotational motion, the axial and transverse deformations and the slopes of the deflections curve of a two-link flexible manipulator modelled by Timoshenko beam theory are considered. By using Hamilton's principle, the dynamic equations of joint angles, vibrations of the flexible links and boundary conditions are derived. Subsequently, some effects of the rotary inertia, shear deformation, axial displacement and flexibility are neglected to obtain all the other dynamic models. Since the tip of the flexible manipulator is in contact with a given constrained surface, a constrained equation should be satisfied whenever the manipulator rotates. In addition, a reaction force composed of the generalized normal and friction forces is generated along the constrained curve.

2. TWO-LINK FLEXIBLE MANIPULATOR

A two-link flexible manipulator in the horizontal plane is shown in Figure 1. One assumes that both the first and the second links are flexible. The ith link has length l_i, uniform mass density ρ_i per unit length, cross-sectional area A_i, and uniform flexural rigidity E_iI_i. The first flexible link is clamped on the rotor of the first motor. The second motor, which is attached at the tip of the first link, can be regarded as a concentrated mass m_1. The second link is clamped on the rotor of the second motor at one end and has a concentrated mass m_2 at the other end

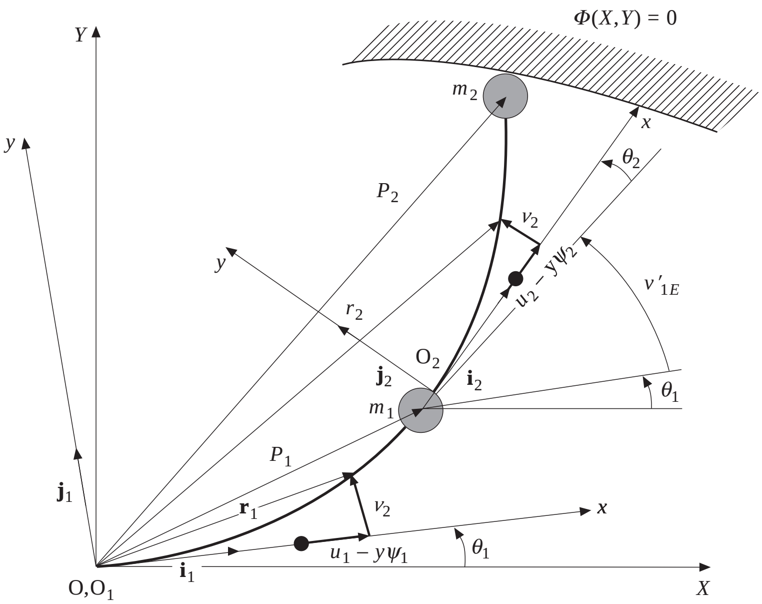

Figure 1. Model of a two-link flexible manipulator.

which is in contact with the rigid constrained surface. Let J_i be the moment of inertia of rotor of the motor i and τ_i be the torque developed by the motor.

2.1. TIMOSHENKO BEAM THEORY

The displacement field of the deformed Timoshenko beam is shown in Figure 1. Let (X, Y) designate an inertia Cartesian co-ordinate variable in the fixed co-ordinate system (OXY) and $[\mathbf{i}_1, \mathbf{j}_1]$ and $[\mathbf{i}_2, \mathbf{j}_2]$ are the orthogonal unit vectors of the moving co-ordinates with origins at O_1 and O_2, respectively. $u_i(x, t)$ and $v_i(x, t)$ represent the axial and transverse displacements of link i at time t and at a spatial point x $(0 < x < l_i)$, respectively, and ψ_i is the slope of the defection curve of the ith link due to bending deformation alone. Let $u_{iE}(t)$ $(=u_i(l_i, t))$ and $v_{iE}(t)$ $(=v_i(l_i, t))$ denote the displacements at the end of link i. For the sake of convenience the following differential notations are used: $(\,)' = \partial(\,)/\partial x$ and $(\dot{\,}) = \partial(\,)/\partial t$.

Let one assume that the elastic deformations $v_i(x, t)$ are small compared to the link lengths. The value of $v'_{1E}(t)$ is so small that the tip angle $\alpha = \tan^{-1} v'_{1E}$ of the first link caused by elastic deformation can be regarded as $\alpha = v'_{1E}$. Let θ_1 and θ_2 be the angles of rotation of motors 1 and 2, respectively. Relations in conjunction with the orthogonal unit vectors $[\mathbf{i}_1, \mathbf{j}_1]$ and $[\mathbf{i}_2, \mathbf{j}_2]$ are

$$\mathbf{i}_1 = [\cos\theta_1, \sin\theta_1]^{\mathrm{T}}, \qquad \mathbf{j}_1 = [-\sin\theta_1, \cos\theta_1]^{\mathrm{T}},$$

$$\mathbf{i}_2 = [\cos(\theta_1 + v'_{1E} + \theta_2), \sin(\theta_1 + v'_{1E} + \theta_2)]^{\mathrm{T}},$$

$$\mathbf{j}_2 = [-\sin(\theta_1 + v'_{1E} + \theta_2), \cos(\theta_1 + v'_{1E} + \theta_2)]^{\mathrm{T}}. \tag{1}$$

Let $\mathbf{P}_i$ and $\mathbf{r}_i$ be the position vectors of the end point and the general point of the flexible link i respectively. The position vectors $\mathbf{P}_i$ and $\mathbf{r}_i$ and their time derivatives are given by

$$\mathbf{P}_1 = (l_1 + u_{1E})\mathbf{i}_1 + v_{1E}\mathbf{j}_1, \qquad \mathbf{r}_1 = (x + u_1 - y\psi_1)\mathbf{i}_1 + (y + v_1)\mathbf{j}_1,$$

$$\mathbf{P}_2 = \mathbf{P}_1 + (l_2 + u_{2E})\mathbf{i}_2 + v_{2E}\mathbf{j}_2, \qquad \mathbf{r}_2 = \mathbf{P}_1 + (x + u_2 - y\psi_2)\mathbf{i}_2 + (y + v_2)\mathbf{j}_2,$$

$$\dot{\mathbf{P}}_1 = (\dot{u}_{1E} - v_{1E}\dot{\theta}_1)\mathbf{i}_1 + (\dot{v}_{1E} + \dot{\theta}_1(l_1 + u_{1E}))\mathbf{j}_1,$$

$$\dot{\mathbf{r}}_1 = ((\dot{u}_1 - y\dot{\psi}_1) - \dot{\theta}_1(y + v_1))\mathbf{i}_1 + (\dot{v}_1 + \dot{\theta}_1(x + u_1 - y\psi_1))\mathbf{j}_1,$$

$$\dot{\mathbf{P}}_2 = \dot{\mathbf{P}}_1 + (\dot{u}_{2E} - v_{2E}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{v}'_{1E}))\mathbf{i}_2 + (\dot{v}_{2E} + (\dot{\theta}_1 + \dot{\theta}_2 + \dot{v}'_{1E})(l_2 + u_{2E}))\mathbf{j}_2,$$

$$\dot{\mathbf{r}}_2 = \dot{\mathbf{P}}_1 + \{(\dot{u}_2 - y\dot{\psi}_2) - (\dot{\theta}_1 + \dot{\theta}_2 + \dot{v}'_{1E})(y + v_2)\}\mathbf{i}_2$$
$$+ \{\dot{v}_2 + (\dot{\theta}_1 + \dot{\theta}_2 + \dot{v}'_{1E})(x + u_2 - y\psi_2)\}\mathbf{j}_2, \qquad (2)$$

where Timoshenko beam theory is employed.

The total kinetic energy T and the potential energy U of the manipulators can be expressed as, respectively,

$$T = \tfrac{1}{2}J_1\dot{\theta}_1^2 + \tfrac{1}{2}J_2(\dot{\theta}_1 + \dot{v}'_{1E} + \dot{\theta}_2)^2 + \sum_{i=1}^{2} \tfrac{1}{2}m_i \dot{\mathbf{P}}_i^{\mathrm{T}} \dot{\mathbf{P}}_i + \sum_{i=1}^{2} \tfrac{1}{2} \int_0^{l_i} \rho_i \dot{\mathbf{r}}_i^{\mathrm{T}} \dot{\mathbf{r}}_i \, \mathrm{d}x, \qquad (3)$$

$$U = \sum_{i=1}^{2} \tfrac{1}{2} \int_0^{l_i} [E_iA_i(u'_i + \tfrac{1}{2}v_i'^2)^2 + K_iG_iA_i(v'_i - \psi_i)^2 + E_iI_i\psi_i'^2] \, \mathrm{d}x. \qquad (4)$$

where K_i is the shear deformation coefficient and G_i is the shear modulus of elasticity. The geometric non-linearity is included in the strain energy.

The virtual works done by the external torques τ_i applied on the links are

$$\delta W = \sum_{i=1}^{2} \tau_i \delta\theta_i. \qquad (5)$$

The second flexible link is in contact with the constrained curve at the tip end. $\mathbf{P}_2 = (X_p, Y_p)^{\mathrm{T}}$ denotes the position vector of the end point of the second link. Hence,

$$X_p = (l_1 + u_{1E}) \cos\theta_1 - v_{1E} \sin\theta_1 + (l_2 + u_{2E}) \cos(\theta_1 + v'_{1E} + \theta_2)$$
$$- v_{2E} \sin(\theta_1 + v'_{1E} + \theta_2),$$
$$Y_p = (l_1 + u_{1E}) \sin\theta_1 + v_{1E} \cos\theta_1 + (l_2 + u_{2E}) \sin(\theta_1 + v'_{1E} + \theta_2)$$
$$- v_{2E} \cos(\theta_1 + v'_{1E} + \theta_2). \qquad (6)$$

The most common methods of treating geometrically contact problems are based on the Lagrange multiplier method. Meanwhile, the geometric condition is enforced by augmenting the Lagrange multiplier as additional system variables.

One assumes that the tip end is always in contact with the constrained surface, which can be described as

$$\Phi(X, Y) = 0. \tag{7}$$

Substituting (6) into (7), the constrained condition has the form

$$\Phi(\theta_1, \theta_2, u_{1E}, u_{2E}, v_{1E}, v'_{1E}, v_{2E}) = 0. \tag{8}$$

Because the end point of the second beam is constrained, a reaction force F^c is generated along the normal direction of the constrained curve, which is calculated by means of two terms [13]. The first one is the product of the scalar Lagrange multiplier with the gradient of the constraint surface, and represents the generalized normal reaction force F_n, which can be written as

$$F_n = \lambda \nabla \Phi = \lambda \begin{bmatrix} b(\theta_1, \theta_2, u_{1E}, u_{2E}, v_{1E}, v'_{1E}, v_{2E}) \\ c(\theta_1, \theta_2, u_{1E}, u_{2E}, v_{1E}, v'_{1E}, v_{2E}) \end{bmatrix}, \tag{9}$$

The second represents the generalized friction force, which accounts for the dry friction force and can be written as

$$F_f = \lambda \text{sign}\,(\lambda)\mu \begin{bmatrix} c(\theta_1, \theta_2, u_{1E}, u_{2E}, v_{1E}, v'_{1E}, v_{2E}) \\ -b(\theta_1, \theta_2, u_{1E}, u_{2E}, v_{1E}, v'_{1E}, v_{2E}) \end{bmatrix}, \tag{10}$$

where λ is the Lagrange multiplier, $\nabla\Phi$ is the gradient of the constraint surface, μ is the coefficient of dry friction and

$$\begin{aligned} b(\theta_1, \theta_2, u_{1E}, u_{2E}, v_{1E}, v'_{1E}, v_{2E}) &= \partial\Phi/\partial X|_{(X_P, Y_P)}, \\ c(\theta_1, \theta_2, u_{1E}, u_{2E}, v_{1E}, v'_{1E}, v_{2E}) &= \partial\Phi/\partial Y|_{(X_P, Y_P)}. \end{aligned} \tag{11}$$

It is seen that F_f is perpendicular to F_n. The sign of the Lagrange multiplier decides if the generalized normal force is directed along the positive or negative normal to the constraint surface. Thus, the constraint force F^c can be expressed as

$$F^c = F_n + F_f = \begin{bmatrix} \lambda b + \lambda \text{ sign}\,(\lambda)\mu c \\ \lambda c - \lambda \text{ sign}\,(\lambda)\mu b \end{bmatrix}. \tag{12}$$

The virtual work done by the constraint forces is

$$\begin{aligned} \delta W^c = F^{c^T}\delta \mathbf{P}_2 &= \lambda[(b + c\mu \text{ sign}\,(\lambda)) \cos\theta_1 + (c - b\mu \text{ sign}\,(\lambda)) \sin\theta_1]\delta u_{1E} \\ &- \lambda[(b + c\mu \text{ sign}\,(\lambda)) \sin\theta_1 - (c - b\mu \text{ sign}\,(\lambda)) \cos\theta_1]\delta v_{1E} \\ &+ \lambda[(b + c\mu \text{ sign}\,(\lambda)) \cos(\theta_1 + v'_{1E} + \theta_2) \\ &+ (c - b\mu \text{ sign}\,(\lambda)) \sin(\theta_1 + v'_{1E} + \theta_2)]\delta u_{2E} \\ &- \lambda[(b + c\mu \text{ sign}\,(\lambda)) \sin(\theta_1 + v'_{1E} + \theta_2) \\ &- (c - b\mu \text{ sign}\,(\lambda)) \cos(\theta_1 + v'_{1E} + \theta_2)]\delta v_{2E} \\ &- \lambda\{(b + c\mu \text{ sign}\,(\lambda))[(l_1 + u_{1E}) \sin\theta_1 + v_{1E} \cos\theta_1 \end{aligned}$$

$$
\begin{aligned}
&+ (l_2 + u_{2E}) \sin(\theta_1 + v'_{2E} + \theta_2) \\
&+ v_{2E} \cos(\theta_1 + v'_{2E} + \theta_2)] - (c - b\mu \operatorname{sign}(\lambda))[(l_1 + u_{1E}) \cos\theta_1 - v_{1E} \sin\theta_1 \\
&+ (l_2 + u_{2E}) \cos(\theta_1 + v'_{2E} + \theta_2) - v_{2E} \sin(\theta_1 + v'_{2E} + \theta_2)]\} \, \delta\theta_1 \\
&- \lambda\{(b + c\mu \operatorname{sign}(\lambda))[(l_2 + u_{2E}) \sin(\theta_1 + v'_{2E} + \theta_2) \\
&- v_{2E} \cos(\theta_1 + v'_{2E} + \theta_2)] \\
&+ (c - b\mu \operatorname{sign}(\lambda))[(l_2 + u_{2E}) \cos(\theta_1 + v'_{2E} + \theta_2) \\
&- v_{2E} \sin(\theta_1 + v'_{2E} + \theta_2)]\} \, \delta\theta_2 \\
&- \lambda\{(b + c\mu \operatorname{sign}(\lambda))[(l_2 + u_{2E}) \sin(\theta_1 + v'_{2E} + \theta_2) \\
&- v_{2E} \cos(\theta_1 + v'_{2E} + \theta_2)] \\
&+ (c - b\mu \operatorname{sign}(\lambda))[(l_2 + u_{2E}) \cos(\theta_1 + v'_{2E} + \theta_2) \\
&- v_{2E} \sin(\theta_1 + v'_{2E} + \theta_2)]\} \, \delta v'_{1E}. \qquad (13)
\end{aligned}
$$

Hamilton's principle for the two-link flexible manipulator is

$$0 = \int_{t_1}^{t_2} (\delta T - \delta U + \delta W + \delta W^c) \, \mathrm{d}t. \qquad (14)$$

Substituting equations (3–5), and (13) into equation (14), one obtains governing equations and boundary conditions of the system. The governing equations of the flexible link 1 are

$$u_1\colon \rho_1 A_1 (x\dot\theta_1^2 + u_1\dot\theta_1^2 - \ddot u_1 + v_1\ddot\theta_1 + 2\dot v_1\dot\theta_1) + E_1 A_1 (u_1'' + v_1' v_1'') = 0, \qquad (15)$$

$$
\begin{aligned}
v_1\colon \rho_1 A_1 (v_1\dot\theta_1^2 - 2\dot u_1\dot\theta_1 - \ddot v_1 - x\ddot\theta_1 - u_1\dot\theta_1) + E_1 A_1 [(u_1'' + v_1' v_1'') v_1' \\
+ (u_1' + \tfrac{1}{2} v_1'^2) v_1''] + K_1 G_1 A_1 (v_1'' - \psi_1') = 0, \qquad (16)
\end{aligned}
$$

$$\psi_1\colon \rho_1 I_1 (\psi_1\dot\theta_1^2 - \ddot\psi_1 - \ddot\theta_1) + K_1 G_1 A_1 (v_1' - \psi_1) + E_1 I_1 \psi_1'' = 0, \qquad (17)$$

and the boundary conditions are

$$u_1(0, t) = 0, \qquad v_1(0, t) = 0, \qquad \psi_1'(0, t) = 0, \qquad \psi_1'(l_1, t) = 0, \quad \text{(18a–d)}$$

$$
\begin{aligned}
m_1 (l_1\dot\theta_1^2 &+ u_{1E}\dot\theta_1^2 - \ddot u_{1E} + v_{1E}\ddot\theta_1 + 2\dot v_{1E}\dot\theta_1) + m_2 (l_1\dot\theta_1^2 + u_{1E}\dot\theta_1^2 - \ddot u_{1E} \\
&+ v_{1E}\ddot\theta_1 + 2\dot v_{1E}\dot\theta_1) - E_1 A_1 (u_{1E}' + \tfrac{1}{2} v_{1E}'^{2\prime\prime}) + E_2 A_2 (u_2'(0, t) + \tfrac{1}{2} v_2'^2(0, t)) \\
&+ \int_0^{l_2} \rho_2 A_2 (l_1\dot\theta_1^2 + u_{1E}\dot\theta_1^2 - \ddot u_{1E} + 2\dot v_{1E}\dot\theta_1 + v_{1E}\ddot\theta_1) \, \mathrm{d}x_1 \\
&+ \lambda[(b + c\mu \operatorname{sign}(\lambda)) \cos\theta + (c - b\mu \operatorname{sign}(\lambda)) \sin\theta_2] = 0, \qquad (19)
\end{aligned}
$$

$$m_1(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1 - 2\dot{u}_{1E}\dot{\theta}_1 - u_{1E}\ddot{\theta}_1) + m_2(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1 - 2\dot{u}_{1E}\dot{\theta}_1 - u_{1E}\ddot{\theta}_1) - E_1A_1(u'_{1E} + \tfrac{1}{2}v'^2_{1E})v'_{1E} - K_1G_1A_1(v'_{1E} - \psi_{1E}) - E_1I_1\psi''_{1E} + E_2I_2\psi''_2(0, t) + \int_0^{l_2} \rho_2A_2(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1 - 2\dot{u}_{1E}\dot{\theta}_1 - u_{1E}\ddot{\theta}_1)\,\mathrm{d}x - \lambda[(b + c\mu\ \mathrm{sign}\ (\lambda))\sin\theta_1 - (c - b\mu\ \mathrm{sign}\ (\lambda))\cos\theta_2] = 0. \tag{20}$$

The governing equations of the flexible link 2 are

$$u_2\colon \rho_2A_2(x\dot{\theta}_1^2 + 2u_2\dot{\theta}_1\dot{v}'_{1E} + 2x\dot{\theta}_1\dot{v}'_{1E} + 2\dot{v}_2\dot{\theta}_1 + 2\dot{v}_2\dot{v}'_{1E} + 2\dot{v}_2\dot{\theta}_2 + u_2\dot{\theta}_1^2 + x\dot{v}'^2_{1E} + u_2\dot{v}'^2_{1E} + 2x\dot{\theta}_1\dot{\theta}_2 + 2u_2\dot{\theta}_1\dot{\theta}_2 + 2x\dot{\theta}_2\dot{v}'_{1E} + 2u_2\dot{\theta}_2\dot{v}'_{1E} + x\dot{\theta}_2^2 + u_2\dot{\theta}_2^2 - \ddot{u}_2 + v_2\ddot{\theta}_1 + v_2\ddot{v}'_{1E} + v_2\ddot{\theta}_2) + E_2A_2(u''_2 + v'_2v''_2) = 0, \tag{21}$$

$$v_2\colon \rho_2A_2(v_2\dot{\theta}_1^2 + 2v_2\dot{\theta}_1\dot{v}'_{1E} + 2v_2\dot{\theta}_1\dot{\theta}_2 + v_2\dot{v}'^2_{1E} + 2v_2\dot{\theta}_2\dot{v}'_{1E} + v_2\dot{\theta}_2^2 - 2\dot{u}_2\dot{\theta}_1 - 2\dot{u}_2\dot{v}'_{1E} - 2\dot{u}_2\dot{\theta}_2 - \ddot{v}_2 - x\ddot{\theta}_1 - u_2\ddot{\theta}_1 - x\ddot{v}'_{1E} - u_2\ddot{v}'_{1E} - x\ddot{\theta}_2 - u_2\ddot{\theta}_2) + E_2A_2[(u''_2 + v'_2v''_2)v'_2 + (u'_2 + \tfrac{1}{2}v'^2_2)v''_2] + K_2G_2A_2(v''_2 - \psi'_2) = 0, \tag{22}$$

$$\psi_2\colon \rho_2I_2(2\psi_2\dot{\theta}_1\dot{v}'_{1E} + 2\psi_2\dot{\theta}_1\dot{\theta}_2 + 2\psi_2\dot{\theta}_2\dot{v}'_{1E} + \psi_2\dot{\theta}_1^2 + \psi_2\dot{v}'^2_{1E} + \psi_2\dot{\theta}_2^2 - \ddot{\psi}_2 - \ddot{\theta}_1 - \ddot{v}'_{1E} - \ddot{\theta}_2) + K_2G_2A_2(v'_2 - \psi_2) + E_2I_2\psi''_2 = 0, \tag{23}$$

and boundary conditions are

$$u_2(0, t) = 0, \qquad v_2(0, t) = 0, \qquad \psi'_2(0, t) = 0, \qquad \psi'_2(l_2, t) = 0, \tag{24a–d}$$

$$m_2(2\dot{v}_{2E}\dot{\theta}_1 + 2\dot{v}_{2E}\dot{v}'_{1E} + 2\dot{v}_{2E}\dot{\theta}_2 + l_2\dot{\theta}_1^2 + 2l_2\dot{\theta}_1\dot{v}'_{1E} + 2l_2\dot{\theta}_1\dot{\theta}_2 + u_{2E}\dot{\theta}_1^2 + 2u_{2E}\dot{\theta}_1\dot{v}'_{1E} + 2u_{2E}\dot{\theta}_1\dot{\theta}_2 + l_2\dot{v}'^2_{1E} + 2l_2\dot{\theta}_2\dot{v}'_{1E} + u_{2E}\dot{v}'^2_{1E} + 2u_{2E}\dot{\theta}_2\dot{v}'_{1E} + l_2\dot{\theta}_2^2 + u_{2E}\dot{\theta}_2^2 - \ddot{u}_{2E} + v_{2E}\ddot{\theta}_1 + v_{2E}\ddot{v}'_{1E} + v_{2E}\ddot{\theta}_2) - E_2A_2(u'_{2E} + \tfrac{1}{2}v'^2_{2E}) + \lambda[(b + c\mu\ \mathrm{sign}\ (\lambda))\cos(\theta_1 + v'_{1E} + \theta_2) + (c - b\mu\ \mathrm{sign}\ (\lambda))\sin(\theta_1 + v'_{1E} + \theta_2) = 0, \tag{25}$$

$$m_2(v_{2E}\dot{\theta}_1^2 - 2\dot{u}_{2E}\dot{\theta}_1 - 2\dot{u}_{2E}\dot{v}'_{1E} - 2\dot{u}_{2E}\dot{\theta}_2 + 2v_{2E}\dot{\theta}_1\dot{v}'_{1E} + 2v_{2E}\dot{\theta}_1\dot{\theta}_2 + v_{2E}\dot{v}'^2_{1E} + 2v_{2E}\dot{\theta}_2\dot{v}'_{1E} + v_{2E}\dot{\theta}_2^2 - \ddot{v}_{2E} - l_2\ddot{\theta}_1 - u_{2E}\ddot{\theta}_1 - l_2\ddot{v}'_{1E} - u_{2E}\ddot{v}'_{1E} - l_2\ddot{\theta}_2 - u_{2E}\ddot{\theta}_2) - E_2A_2(u'_{2E} + \tfrac{1}{2}v'^2_{2E})v'_{2E} - K_2G_2A_2(v'_{2E} - \psi_{2E}) - \lambda[(b + c\mu\ \mathrm{sign}\ (\lambda))\sin(\theta_1 + v'_{1E} + \theta_2) - (c - b\mu\ \mathrm{sign}\ (\lambda)) \times \cos(\theta_1 + v'_{1E} + \theta_2)] = 0. \tag{26}$$

The governing equation of joint angle θ_1 is

$$\int_0^{l_1} [\rho_1A_1(\ddot{u}_1v_1 - v_1^2\ddot{\theta}_1 - 2v_1\dot{v}_1\dot{\theta}_1 - x\ddot{v}_1 - u_1\ddot{v}_1 - x^2\ddot{\theta}_1 - 2xu_1\ddot{\theta}_1 - 2x\dot{u}_1\dot{\theta}_1 - u_1^2\ddot{\theta}_1 - 2u_1\dot{u}_1\dot{\theta}_1) - \rho_1I_1(\ddot{\psi}_1 + \ddot{\theta}_1 + \psi_1^2\ddot{\theta}_1 + 2\psi_1\dot{\psi}_1\dot{\theta}_1)]\,\mathrm{d}x$$

$$
\begin{aligned}
&- \int_0^{l_2} [\rho_2 A_2(2x\dot{u}_2\dot{v}'_{1E} + 2xu_2\ddot{v}'_{1E} + 2x\dot{u}_2\dot{\theta}_2 + 2xu_2\ddot{\theta}_2 + x^2\ddot{\theta}_1 + 2x\dot{u}_2\dot{\theta}_1 \\
&+ 2xu_2\ddot{\theta}_1 + 2u_2\dot{u}_2\dot{v}'_{1E} + u_2^2\ddot{v}'_{1E} + x^2\ddot{\theta}_2 + 2u_2\dot{u}_2\dot{\theta}_2 + u_2^2\ddot{\theta}_2 + 2l_1\dot{u}_{1E}\dot{\theta}_1 \\
&+ 2l_1u_{1E}\ddot{\theta}_1 - \ddot{u}_{1E}v_{1E} + 2v_{1E}\dot{v}_{1E}\dot{\theta}_1 + v_{1E}^2\ddot{\theta}_1 + l_1\ddot{v}_{1E} + u_{1E}\ddot{v}_{1E} + l_1^2\ddot{\theta}_1 \\
&+ 2u_{1E}\dot{u}_{1E}\dot{\theta}_1 + u_{1E}^2\ddot{\theta}_1 - \ddot{u}_2v_2 + 2v_2\dot{v}_2\dot{\theta}_1 + v_2^2\ddot{\theta}_1 + 2v_2\dot{v}_2\dot{v}'_{1E} + v_2^2\ddot{v}'_{1E} \\
&+ 2v_2\dot{v}_2\dot{\theta}_2 + v_2^2\ddot{\theta}_2 + x\ddot{v}_2 + u_2\ddot{v}_2 + 2u_2\dot{u}_2\dot{\theta}_1 + u_2^2\ddot{\theta}_1 + x^2\ddot{v}'_{1E}) + \rho_2 I_2(\ddot{\theta}_1 \\
&+ 2\psi_2\dot{\psi}_2\dot{v}'_{1E} + \psi_2^2\ddot{v}'_{1E} + 2\psi_2\dot{\psi}_2\dot{\theta}_2 + \psi_2^2\ddot{\theta}_2 + \ddot{\psi}_2 + \ddot{\theta}_2 + \ddot{v}'_{1E} + 2\psi_2\dot{\psi}_2\dot{\theta}_1 \\
&+ \psi_2^2\ddot{\theta}_1)]\,\mathrm{d}x \\
&- m_1(2v_{1E}\dot{v}_{1E}\dot{\theta}_1 + v_{1E}^2\ddot{\theta}_1 - \ddot{u}_{1E}v_{1E} + l_1\ddot{v}_{1E} + u_{1E}\ddot{v}_{1E} + l_1^2\ddot{\theta}_1 + 2l_1\dot{u}_{1E}\dot{\theta}_1 + 2l_1u_{1E}\ddot{\theta}_1 \\
&+ v_{2E}^2\ddot{\theta}_1 + 2v_{2E}\dot{v}_{2E}\dot{v}'_{1E} + v_{2E}^2\ddot{v}'_{1E} + 2v_{2E}\dot{v}_{2E}\dot{\theta}_2 + v_{2E}^2\ddot{\theta}_2 + l_2^2\ddot{v}_{2E} + u_{2E}\ddot{v}_{2E} + l_2^2\ddot{\theta}_1 \\
&+ 2l_2\dot{u}_{2E}\dot{\theta}_1 + 2l_2u_{2E}\ddot{\theta}_1 + 2u_{2E}\dot{u}_{2E}\dot{\theta}_1 + u_{2E}^2\ddot{\theta}_1 + l_2^2\ddot{v}'_{1E} + 2l_2\dot{u}_{2E}\dot{v}'_{1E} + 2l_2u_{2E}\ddot{v}'_{1E} \\
&+ 2u_{2E}\dot{u}_{2E}\dot{v}'_{1E} + u_{2E}^2\ddot{v}'_{1E} + l_2^2\ddot{\theta}_2 + 2l_2\dot{u}_{2E}\dot{\theta}_2 + 2l_2u_{2E}\ddot{\theta}_2 + 2u_{2E}\dot{u}_{2E}\dot{\theta}_2 + u_{2E}^2\ddot{\theta}_2) \\
&- J_1\ddot{\theta}_1 - J_2(\ddot{\theta}_1 + \ddot{v}'_{1E} + \ddot{\theta}_2) + \tau_1 - \lambda\{(b + c\mu\ \mathrm{sign}\ (\lambda))[(l_1 + u_{1E})\sin\theta_1 \\
&+ v_{1E}\cos\theta_1 \\
&+ (l_2 + u_{2E})\sin(\theta_1 + v'_{2E} + \theta_2) + v_{2E}\cos(\theta_1 + v'_{2E} + \theta_2)] \\
&- (c - b\mu\ \mathrm{sign}\ (\lambda))[(l_1 + u_{1E})\cos\theta_1 - v_{1E}\sin\theta_1 \\
&+ (l_2 + u_{2E})\cos(\theta_1 + v'_{2E} + \theta_2) - v_{2E}\sin(\theta_1 + v'_{2E} + \theta_2)]\} = 0. \qquad (27)
\end{aligned}
$$

The governing equation of joint angle θ_2 is

$$
\begin{aligned}
&\int_0^{l_2} [\rho_2 A_2(\ddot{u}_2v_2 - 2x\dot{u}_2\dot{v}'_{1E} - 2xu_2\ddot{v}'_{1E} - 2x\dot{u}_2\dot{\theta}_2 - 2xu_2\ddot{\theta}_2 - x^2\ddot{\theta}_1 - 2x\dot{u}_2\dot{\theta}_1 \\
&- 2xu_2\ddot{\theta}_1 - 2u_2\dot{u}_2\dot{v}'_{1E} - u_2^2\ddot{v}'_{1E} - x^2\ddot{\theta}_2 - 2u_2\dot{u}_2\dot{\theta}_2 - u_2^2\ddot{\theta}_2 - 2v_2\dot{v}_2\dot{\theta}_1 - v_2^2\ddot{\theta}_1 \\
&- 2v_2\dot{v}_2\dot{v}'_{1E} - v_2^2\ddot{v}'_{1E} - 2v_2\dot{v}_2\dot{\theta}_2 - v_2^2\ddot{\theta}_2 - x\ddot{v}_2 - u_2\ddot{v}_2 - 2u_2\dot{u}_2\dot{\theta}_1 - u_2^2\ddot{\theta}_1 - x^2\ddot{v}'_{1E})] \\
&- \rho_2 I_2(\ddot{\theta}_1 + 2\psi_2\dot{\psi}_2\dot{v}'_{1E} + \psi_2^2\ddot{v}'_{1E} + 2\psi_2\dot{\psi}_2\dot{\theta}_2 + \psi_2^2\ddot{\theta}_2 + \ddot{\psi}_2 + \ddot{\theta}_2 + \ddot{v}'_{1E} \\
&+ 2\psi_2\dot{\psi}_2\dot{\theta}_1 + \psi_2^2\ddot{\theta}_1)]\,dx - m_2(2v_{2E}\dot{v}_{2E}\dot{\theta}_1 + v_{2E}^2\ddot{\theta}_1 - \ddot{u}_{2E}v_{2E} + 2v_{2E}\dot{v}_{2E}\dot{v}'_{1E} \\
&+ v_{2E}^2\ddot{v}'_{1E} + 2v_{2E}\dot{v}_{2E}\dot{\theta}_2 + v_{2E}^2\ddot{\theta}_2 + l_2\ddot{v}_{2E} + u_{2E}\ddot{v}_{2E} + l_2^2\ddot{\theta}_1 + 2l_2\dot{u}_{2E}\dot{\theta}_1 \\
&+ 2l_2u_{2E}\ddot{\theta}_1 + 2u_{2E}\dot{u}_{2E}\dot{\theta}_1 + u_{2E}^2\ddot{\theta}_1 + l_2^2\ddot{v}'_{1E} + 2l_2\dot{u}_{2E}\dot{v}'_{1E} + 2l_2u_{2E}\ddot{v}'_{1E} \\
&+ 2u_{2E}\dot{u}_{2E}\dot{v}'_{1E} + u_{2E}^2\ddot{v}'_{1E} + l_2^2\ddot{\theta}_2 + 2l_2\dot{u}_{2E}\dot{\theta}_2 + 2l_2u_{2E}\ddot{\theta}_2 + 2u_{2E}\dot{u}_{2E}\dot{\theta}_2 + u_{2E}^2\ddot{\theta}_2) \\
&- J_2(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{v}'_{1E}) + \tau_2 - \lambda\{(b + c\mu\ \mathrm{sign}\ (\lambda))[(l_2 + u_{2E})\sin(\theta_1 + v'_{2E} + \theta_2) \\
&- v_{2E}\cos(\theta_1 + v'_{2E} + \theta_2)] + (c - b\mu\ \mathrm{sign}\ (\lambda))[(l_2 + u_{2E})\cos(\theta_1 + v'_{2E} + \theta_2) \\
&- v_{2E}\sin(\theta_1 + v'_{2E} + \theta_2)]\} = 0. \qquad (28)
\end{aligned}
$$

The non-linear partial differential equations, (15)–(17), (21)–(23), include the second order spatial and time derivatives of all the variables u_i, v_i and ψ_i ($i = 1, 2$). The six boundary conditions, (18a–d), (19), (20), (24a–d), (25) and (26), are satisfied to solve those equations. Equations (27) and (28) are the second order time derivatives of θ_1 and θ_2 respectively and describe the rotational motions of the two links. Equations (15–17) describe the flexural vibrations of u_1, v_1 and ψ_1 of link 1 while equations (21–23) describe the flexural vibrations of u_2, v_2 and ψ_2 of link 2. It is seen that the rigid body motion and flexural vibration are coupled. The boundary conditions (18a, b) and (24a, b) indicate that the flexural links are clamped. The boundary conditions (18c, d) and (24c, d) state that zero moment exists at the end point. The boundary conditions (19), (20), (25) and (26) represent the force equilibriums at the end points of the flexible links along the $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{i}_2$ and $\mathbf{j}_2$ directions respectively.

2.2. EULER BEAM THEORY

If the slenderness of the beam is very small, Euler beam theory can be used to describe the motion of the manipulator by setting $\psi_1 = v_1'$ and $\psi_2 = v_2'$ and neglecting the rotating inertia effects of $\rho_1 I_1(\psi_1\dot{\theta}_1^2 - \ddot{\psi}_1 - \ddot{\theta}_1)$ and $\rho_2 I_2 \quad (2\psi_2\dot{\theta}_1\dot{v}_{1E}' + 2\psi_2\dot{\theta}_1\dot{\theta}_2 + \psi_2\dot{v}_{1E}'^2 + 2\psi_2\dot{\theta}_2\dot{v}_{1E}' + \psi_2\dot{\theta}_1^2 + \psi_2\dot{\theta}_2^2 - \ddot{\psi}_2 - \ddot{\theta}_1 - \ddot{v}_{1E}' - \ddot{\theta}_2)$. One obtains the governing equations of the flexible link 1:

u_1: equation (15),

$$v_1\colon \rho_1 A_1(v_1\dot{\theta}_1^2 - \ddot{v}_1 - x\ddot{\theta}_1 - u_1\ddot{\theta}_1 - 2\dot{u}_1\dot{\theta}_1) + E_1A_1[(u_1'' + v_1'v_1'')v_1' + (u_1' + \tfrac{1}{2}v_1'^2)v_1''] - E_1I_1v_1'''' = 0, \tag{29}$$

and boundary conditions:

$$u_1(0, t) = 0, \qquad v_1(0, t) = 0, \qquad v_1''(0, t) = 0, \qquad v_1''(l_1, t) = 0, \quad \text{(30a–d)}$$

$$m_1(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1 - 2\dot{u}_{1E}\dot{\theta}_1 - u_{1E}\ddot{\theta}_1) + m_2(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1 - 2\dot{u}_{1E}\dot{\theta}_1 - u_{1E}\ddot{\theta}_1) - E_1A_1(u_{1E}' + \tfrac{1}{2}v_{1E}'^2)v_{1E}' - E_1I_1v_{1E}''' + E_2I_2v_2'''(0, t) + \int_0^{l_2} \rho_2A_2(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1 - 2\dot{u}_{1E}\dot{\theta}_1 - u_{1E}\ddot{\theta}_1)\,\mathrm{d}x - \lambda[(b + c\mu\,\mathrm{sign}\,(\lambda))\sin\theta_1 - (c - b\mu\,\mathrm{sign}\,(\lambda))\cos\theta_2] = 0, \tag{31}$$

and equation (19).

The governing equations of the flexible link 2 are

u_2: equation (21),

$$v_2\colon \rho_2A_2(v_2\dot{\theta}_1^2 + 2v_2\dot{\theta}_1\dot{v}_{1E}' + 2v_2\dot{\theta}_1\dot{\theta}_2 + v_2\dot{v}_{1E}'^2 + 2v_2\dot{\theta}_2\dot{v}_{1E}' + v_2\dot{\theta}_2^2 - 2\dot{u}_2\dot{\theta}_1 - \ddot{v}_2 - x\ddot{\theta}_1 - u_2\ddot{\theta}_1 - x\ddot{v}_{1E}' - u_2\ddot{v}_{1E}' - 2\dot{u}_2\dot{v}_{1E}' - x\ddot{\theta}_2 - u_2\ddot{\theta}_2 - 2\dot{u}_2\dot{\theta}_2) + E_2A_2[(u_2'' + v_2'v_2'')v_2' + (u_2' + \tfrac{1}{2}v_2'^2)v_2''] - E_2I_2v_2'''' = 0, \tag{32}$$

and boundary conditions are

$$u_2(0,t)=0,\qquad v_2(0,t)=0,\qquad v_2''(0,t)=0,\qquad v_2''(l_2,t)=0, \tag{33}$$

$$m_2(v_{2E}\dot\theta_1^2-2\dot u_{2E}\dot\theta_1-2\dot u_{2E}\dot v_{1E}'-2\dot u_{2E}\dot\theta_2+2v_{2E}\dot\theta_1\dot v_{1E}'+2v_{2E}\dot\theta_1\dot\theta_2+v_{2E}\dot v_{1E}'^2$$
$$+2v_{2E}\dot\theta_2\dot v_{1E}'+v_{2E}\dot\theta_2^2-\ddot v_{2E}-l_2\ddot\theta_1-u_{2E}\ddot\theta_1-l_2\ddot v_{1E}'$$
$$-u_{2E}\ddot v_{1E}'-l_2\ddot\theta_2-u_{2E}\ddot\theta_2)-E_2A_2(u_{2E}'+\tfrac{1}{2}v_{2E}'^2)v_{2E}'+E_2I_2v_{2E}'''$$
$$-\lambda[(b+c\mu\,\mathrm{sign}\,(\lambda))\sin(\theta_1+v_{1E}'+\theta_2)-(c-b\mu\,\mathrm{sign}\,(\lambda))$$
$$\times\cos(\theta_1+v_{1E}'+\theta_2)]=0, \tag{34}$$

and equation (25).

The governing equation of the joint angle θ_1 is

$$\int_0^{l_1}[\rho_1A_1(\ddot u_1v_1-v_1^2\ddot\theta_1-2v_1\dot v_1\dot\theta_1-x\ddot v_1-u_1\ddot v_1-x^2\ddot\theta_1-2xu_1\ddot\theta_1-2x\dot u_1\dot\theta_1$$
$$-u_1^2\ddot\theta_1-2u_1\dot u_1\dot\theta_1)]\,\mathrm{d}x-\int_0^{l_2}[\rho_2A_2(2x\dot u_2\dot v_{1E}'+2xu_2\ddot v_{1E}'+2x\dot u_2\dot\theta_2+2xu_2\ddot\theta_2$$
$$+x^2\ddot\theta_1+2x\dot u_2\dot\theta_1+2xu_2\ddot\theta_1+2u_2\dot u_2\dot v_{1E}'+u_2^2\ddot v_{1E}'+x^2\ddot\theta_2+2u_2\dot u_2\dot\theta_2+u_2^2\ddot\theta_2$$
$$+2l_1\dot u_{1E}\dot\theta_1+2l_1u_{1E}\ddot\theta_1-\ddot u_{1E}v_{1E}+2v_{1E}\dot v_{1E}\dot\theta_1+v_{1E}^2\ddot\theta_1+l_1\ddot v_{1E}+u_{1E}\ddot v_{1E}$$
$$+l_1^2\ddot\theta_1+2u_{1E}\dot u_{1E}\dot\theta_1+u_{1E}^2\ddot\theta_1-\ddot u_2v_2+2v_2\dot v_2\dot\theta_1+v_2^2\ddot\theta_1+2v_2\dot v_2\dot v_{1E}'$$
$$+v_2^2\ddot v_{1E}'+2v_2\dot v_2\dot\theta_2+v_2^2\ddot\theta_2+x\ddot v_2+u_2\ddot v_2+2u_2\dot u_2\dot\theta_1+u_2^2\ddot\theta_1+x^2\ddot v_{1E}')]\,\mathrm{d}x$$
$$-m_1(2v_{1E}\dot v_{1E}\dot\theta_1+v_{1E}^2\ddot\theta_1-\ddot u_{1E}v_{1E}+l_1\ddot v_{1E}+u_{1E}\ddot v_{1E}+l_1^2\ddot\theta_1+2l_1\dot u_{1E}\dot\theta_1$$
$$+2l_1u_{1E}\ddot\theta_1+2u_{1E}\dot u_{1E}\dot\theta_1+u_{1E}^2\ddot\theta_1)-m_2(2v_{1E}\dot v_{1E}\dot\theta_1+v_{1E}^2\ddot\theta_1-\ddot u_{1E}v_{1E}+l_1\ddot v_{1E}$$
$$+u_{1E}\ddot v_{1E}+l_1^2\ddot\theta_1+2l_1\dot u_{1E}\dot\theta_1+2l_1u_{1E}\ddot\theta_1+2u_{1E}\dot u_{1E}\dot\theta_1+v_{2E}^2\ddot\theta_2+l_2^2\ddot v_{2E}+u_{2E}\ddot v_{2E}$$
$$+l_2^2\ddot\theta_1+2l_2\dot u_{2E}\dot\theta_1+2l_2u_{2E}\ddot\theta_1+2u_{2E}\dot u_{2E}\dot\theta_1+u_{2E}^2\ddot\theta_1+l_2^2\ddot v_{1E}'+2l_2\dot u_{2E}\dot v_{1E}'$$
$$+2l_2u_{2E}\ddot v_{1E}'+2u_{2E}\dot u_{2E}\dot v_{1E}'+u_{2E}^2\ddot v_{1E}'+l_2^2\ddot\theta_2+2l_2\dot u_{2E}\dot\theta_2+2l_2u_{2E}\ddot\theta_2+2u_{2E}\dot u_{2E}\dot\theta_2$$
$$+u_{2E}^2\ddot\theta_2)-J_1\ddot\theta_1-J_2(\ddot\theta_1+\ddot v_{1E}'+\ddot\theta_2)+\tau_1$$
$$-\lambda\{(b+c\mu\,\mathrm{sign}\,(\lambda))[(l_1+u_{1E})\sin\theta_1$$
$$+v_{1E}\cos\theta_1+(l_2+u_{2E})\sin(\theta_1+v_{2E}'+\theta_2)$$
$$+v_{2E}\cos(\theta_1+v_{2E}'+\theta_2)]-(c-b\mu\,\mathrm{sign}\,(\lambda))[(l_1+u_{1E})\cos\theta_1-v_{1E}\sin\theta_1$$
$$+(l_2+u_{2E})\cos(\theta_1+v_{2E}'+\theta_2)-v_{2E}\sin(\theta_1+v_{2E}'+\theta_2)]\}=0, \tag{35}$$

The governing equation of the joint angle θ_2 is

$$\int_0^{l_2} [\rho_2 A_2(\ddot{u}_2 v_2 - 2x\dot{u}_2\dot{v}'_{1E} - 2xu_2\ddot{v}'_{1E} - 2x\dot{u}_2\dot{\theta}_2 - 2xu_2\ddot{\theta}_2 - x^2\ddot{\theta}_1 - 2x\dot{u}_2\dot{\theta}_1$$

$$- 2xu_2\ddot{\theta}_1 - 2u_2\dot{u}_2\dot{v}'_{1E} - u_2^2\ddot{v}'_{1E} - x^2\ddot{\theta}_2 - 2u_2\dot{u}_2\dot{\theta}_2 - u_2^2\ddot{\theta}_2 - 2v_2\dot{v}_2\dot{\theta}_1 - v_2^2\ddot{\theta}_1$$

$$- 2v_2\dot{v}_2\dot{v}'_{1E} - v_2^2\ddot{v}'_{1E} - 2v_2\dot{v}_2\dot{\theta}_2 - v_2^2\ddot{\theta}_2 - x\ddot{v}_2 - u_2\ddot{v}_2 - 2u_2\dot{u}_2\dot{\theta}_1 - u_2^2\ddot{\theta}_1$$

$$- x^2\ddot{v}'_{1E})]\,\mathrm{d}x$$

$$- m_2(2v_{2E}\dot{v}_{2E}\dot{\theta}_1 + v_{2E}^2\ddot{\theta}_1 - \ddot{u}_{2E}v_{2E} + 2v_{2E}\dot{v}_{2E}\dot{v}'_{1E} + v_{2E}^2\ddot{v}'_{1E}$$

$$+ 2v_{2E}\dot{v}_{2E}\dot{\theta}_2 + v_{2E}^2\ddot{\theta}_2 + l_2\ddot{v}_{2E} + u_{2E}\ddot{v}_{2E} + l_2^2\ddot{\theta}_1 + 2l_2\dot{u}_{2E}\dot{\theta}_1 + 2l_2u_{2E}\ddot{\theta}_1$$

$$+ 2u_{2E}\dot{u}_{2E}\dot{\theta}_1 + u_{2E}^2\ddot{\theta}_1 + l_2^2\ddot{v}'_{1E} + 2l_2\dot{u}_{2E}\dot{v}'_{1E} + 2l_2u_{2E}\ddot{v}'_{1E} + 2u_{2E}\dot{u}_{2E}\dot{v}'_{1E} + u_{2E}^2\ddot{v}'_{1E}$$

$$+ l_2^2\ddot{\theta}_2 + 2l_2\dot{u}_{2E}\dot{\theta}_2 + 2l_2u_{2E}\ddot{\theta}_2 + 2u_{2E}\dot{u}_{2E}\dot{\theta}_2 + u_{2E}^2\ddot{\theta}_2) - J_2(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{v}'_{1E}) + \tau_2$$

$$- \lambda\{(b + c\mu\,\mathrm{sign}\,(\lambda))[(l_2 + u_{2E})\sin(\theta_1 + v'_{2E} + \theta_2) - v_{2E}\cos(\theta_1 + v'_{2E} + \theta_2)]$$

$$+ (c - b\mu\,\mathrm{sign}\,(\lambda))[(l_2 + u_{2E})\cos(\theta_1 + v'_{2E} + \theta_2) - v_{2E}$$

$$\times \sin(\theta_1 + v'_{2E} + \theta_2)]\} = 0. \quad (36)$$

By setting $u_i = 0$, $(i = 1, 2)$ in equations (29–36), Matsuno *et al*. [1] derived the same differential equations by using both Hamilton's principle and the equilibriums of forces and moments acting on a differential element.

2.3. SIMPLE FLEXURE MODEL

In the simple flexible model, one eliminates the axial displacements u_1 and u_2 but retains the axially inertia effects [14]. Equations (15), (19), (21) and (25) contain the constrained force, the tip mass and the inertial force of the beam. The reduction process is to carry these effects in the u equations into the v governing equations (29) and (32) and its boundary conditions (31) and (34). Thus, one may define the internal axial forces as

$$p_1(x, t) = E_1A_1(u'_1 + \tfrac{1}{2}v_1'^2), \quad (37)$$

$$p_2(x, t) = E_2A_2(u'_2 + \tfrac{1}{2}v_2'^2), \quad (38)$$

The relationships of the inertia force of the tip mass and the constrained force at the rightside are, respectively,

$$p_1(l_1, t) = m_1(l_1\dot{\theta}_1^2 + v_{1E}\ddot{\theta}_1 + 2\dot{v}_{1E}\dot{\theta}_1) + m_2(l_1\dot{\theta}_1^2 + v_{1E}\ddot{\theta}_1 + 2\dot{v}_{1E1}\dot{\theta}_1)$$

$$+ p_2(0, t)\int_0^{l_2} \rho_2A_2(2\dot{v}_{1E}\dot{\theta}_1 + l_1\dot{\theta}_1^2 + v_{1E}\ddot{\theta}_1)\,\mathrm{d}x_1$$

$$+ \lambda[(b + c\mu\,\mathrm{sign}\,(\lambda))\cos\theta_1 + (c - b\mu\,\mathrm{sign}\,(\lambda))\sin\theta_2], \quad (39)$$

$$p_2(l_2, t) = m_2(\dot{v}_{2E}\dot{\theta}_1 + \dot{v}_{2E}\dot{v}'_{1E} + \dot{v}_{2E}\dot{\theta}_2 + l_2\dot{\theta}_1^2 + 2l_2\dot{\theta}_1\dot{v}'_{1E} + 2l_2\dot{\theta}_1\dot{\theta}_2$$
$$+ l_2\ddot{v}'_{1E} + 2l_2\dot{\theta}_2\dot{v}'_{1E} + l_2\dot{\theta}_2^2\dot{v}'_{1E} + l_2\dot{\theta}_2^2 + \dot{v}_{2E}\dot{\theta}_1 + v_{2E}\ddot{\theta}_1 + \dot{v}_{2E}\dot{v}'_{1E}$$
$$+ v_{2E}\ddot{v}'_{1E} + \dot{v}_{2E}\dot{\theta}_2 + v_{2E}\ddot{\theta}_2) + \lambda[(b + c\mu \text{ sign } (\lambda)) \cos (\theta_1 + v'_{1E} + \theta_2)$$
$$+ (c - b\mu \text{ sign } (\lambda)) \sin (\theta_1 + v'_{1E} + \theta_2). \qquad (40)$$

Neglecting u_i and $\ddot{u}_i$ ($i = 1, 2$) in equations (15) and (21), one has

$$p'_1(x, t) = -\rho_1 A_1(2\dot{v}_1\dot{\theta}_1 + x\dot{\theta}_1^2 + v_1\ddot{\theta}_1), \qquad (41)$$

$$p'_2(x, t) = -\rho_2 A_2(x\dot{\theta}_1^2 + 2x\dot{\theta}_1\dot{v}'_{1E} + \dot{v}_2\dot{\theta}_1 + \dot{v}_2\dot{v}'_{1E} + \dot{v}_2\dot{\theta}_2 + x\dot{v}'^2_{1E} + 2x\dot{\theta}_1\dot{\theta}_2$$
$$+ 2x\dot{\theta}_2\dot{v}'_{1E} + x\dot{\theta}_2^2 + v_2\ddot{\theta}_1 + \dot{v}_2\dot{\theta}_1 + v_2\ddot{v}'_{1E} + \dot{v}_2\dot{v}'_{1E} + v_2\ddot{\theta}_2 + \dot{v}_2\dot{\theta}_2). \qquad (42)$$

As a result, one has

$$p_1(x, t) = p_1(l_1, t) - \int_0^{l_1} \frac{\partial}{\partial x} p_1(x, t)\, \mathrm{d}x$$

$$= m_1(l_1\dot{\theta}_1^2 + v_{1E}\ddot{\theta}_1 + 2\dot{v}_{1E}\dot{\theta}_1) + m_2(l_1\dot{\theta}_1^2 + v_{1E}\ddot{\theta}_1 + 2\dot{v}_{1E1}\dot{\theta}) + p_2(0, t)$$

$$+ \int_0^{l_2} \rho_2 A_2(2\dot{v}_{1E}\dot{\theta}_1 + l_1\dot{\theta}_1^2 + v_{1E}\ddot{\theta}_{11})\, \mathrm{d}x_1 + \lambda[(b + c\mu \text{ sign } (\lambda)) \cos \theta_1$$

$$+ (c - b\mu \text{ sign } (\lambda)) \sin \theta_2] + [\rho_1 A_1(2\dot{v}_1\dot{\theta}_1 + x\dot{\theta}_1^2 + v_1\ddot{\theta}_1)]_x^{l_1}, \qquad (43)$$

$$p_2(x, t) = p_2(l_2, t) - \int_x^{l_2} \frac{\partial}{\partial x} p_2(x, t)\, \mathrm{d}x$$

$$= m_2(\dot{v}_{2E}\dot{\theta}_1 + \dot{v}_{2E}\dot{v}_{1E} + \dot{v}_{2E}\dot{\theta}_2 + l_2\dot{\theta}_1^2 - 2l_2\dot{\theta}_1\dot{v}'_{1E} + 2l_2\dot{\theta}_1\dot{\theta}_2$$
$$+ 2u_{2E}\dot{\theta}_1\dot{\theta}_2 + l_2\ddot{v}'_{1E} + 2l_2\dot{\theta}_2\dot{v}'_{1E} + l_2\dot{\theta}_2^2 + \dot{v}_{2E}\dot{\theta}_1 + v_{2E}\ddot{\theta}_1 + \dot{v}_{2E}\dot{v}'_{1E}$$
$$+ v_{2E}\ddot{v}'_{1E} + \dot{v}_{2E}\dot{\theta}_2 + v_{2E}\ddot{\theta}_2) + \lambda[(b + c\mu \text{ sign } (\lambda)) \cos (\theta_1 + v'_{1E} + \theta_2)$$
$$+ (c - b\mu \text{ sign } (\lambda)) \sin (\theta_1 + v'_{1E} + \theta_2) + [\rho_2 A_2(x\dot{\theta}_1^2 + 2x\dot{\theta}_1\dot{v}'_{1E}$$
$$+ \dot{v}_2\dot{\theta}_1 + \dot{v}_2\dot{v}'_{1E} + \dot{v}_2\dot{\theta}_2 + x\dot{v}'^2_{1E} + 2x\dot{\theta}_1\dot{\theta}_2 + 2x\dot{\theta}_2\dot{v}'_{1E} + x\dot{\theta}_2^2$$
$$+ v_2\ddot{\theta}_1 + \dot{v}_2\dot{\theta}_1 + v_2\ddot{v}'_{1E} + \dot{v}_2\dot{v}'_{1E} + v_2\ddot{\theta}_2 + \dot{v}_2\dot{\theta}_2]_x^{l_2}. \qquad (44)$$

The governing equation (29) of link 1 can be rewritten as

$$\rho_1 A_1(v_1\dot{\theta}_1^2 - \ddot{v}_1 - x\ddot{\theta}_1) - [p_1 v'_1]' - E_1 I_1 v''''_1 = 0, \qquad (45)$$

and boundary conditions are

$$v_1(0, t) = 0, \qquad v''_1(0, t) = 0, \qquad v''_1(l_1, t) = 0, \qquad (46a\text{–}c)$$

$$m_1(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1) + m_2(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1) - p_1(l_1, t) - E_1I_1v_{1E}'''$$
$$+ E_2I_2v_2'''(0, t) + \int_0^{l_2} \rho_2A_2(v_{1E}\dot{\theta}_1^2 - \ddot{v}_{1E} - l_1\ddot{\theta}_1)\,\mathrm{d}x$$
$$- \lambda[(b + c\mu \operatorname{sign}(\lambda)) \sin\theta - (c - b\mu \operatorname{sign}(\lambda))\cos\theta_2] = 0. \tag{47}$$

The governing equation (32) of link 2 can be rewritten as

$$\rho_2A_2(v_2\dot{\theta}_1^2 + 2v_2\dot{\theta}_1\dot{v}_{1E}' + 2v_2\dot{\theta}_1\dot{\theta}_2 + v_2\dot{v}_{1E}'^2 + 2v_2\dot{\theta}_2\dot{v}_{1E}' + v_2\dot{\theta}_2^2 - \ddot{v}_2$$
$$- x\ddot{\theta}_1 - x\ddot{v}_{1E}' - x\ddot{\theta}_2) - [p_2v_2'] - E_2I_2v_2'''' = 0, \tag{48}$$

and boundary conditions are

$$v_2(0, t) = 0, \qquad v_2''(0, t) = 0, \qquad v_2''(l_2, t) = 0, \tag{49a–c}$$

$$m_2(v_{2E}\dot{\theta}_1^2 + 2v_{2E}\dot{\theta}_1\dot{v}_{1E}' + 2v_{2E}\dot{\theta}_1\theta_2 + v_{2E}\dot{v}_{1E}'^2 + 2v_{2E}\dot{v}_{1E}' + v_{2E}\dot{\theta}_2^2$$
$$- \ddot{v}_{2E} - l_2\ddot{\theta}_1 - l_2\ddot{v}_{1E}' - l_2\ddot{\theta}_2) - p_2(l_2, t) + E_2I_2v_{2E}'''$$
$$- \lambda[(b + c\mu \operatorname{sign}(\lambda)) \sin(\theta_1 + v_{1E}' + \theta_2)$$
$$- (c - b\mu \operatorname{sign}(\lambda))\cos(\theta_1 + v_{1E}' + \theta_2)] = 0. \tag{50}$$

The governing equation of the joint angle θ_1 is

$$\int_0^{l_1} [\rho_1A_1(-v_1^2\ddot{\theta}_1 - 2v_1\dot{v}_1\dot{\theta}_1 - x\ddot{v}_1 - x^2\ddot{\theta}_1)]\,\mathrm{d}x - \int_0^{l_2} [\rho_2A_2(x^2\ddot{\theta}_1 + x^2\ddot{\theta}_2$$
$$+ 2v_{1E}\dot{v}_{1E}\dot{\theta}_1 + v_{1E}^2\ddot{\theta}_1 + l_1\ddot{v}_{1E} + l_1^2\ddot{\theta}_1 + 2v_2\dot{v}_2\dot{\theta}_1 + v_2^2\ddot{\theta}_1 + 2v_2\dot{v}_2\dot{v}_{1E}'$$
$$+ v_2^2\ddot{v}_{1E}' + 2v_2\dot{v}_2\dot{\theta}_2 + v_2^2\ddot{\theta}_2 + x\ddot{v}_2 + x^2\ddot{v}_{1E}')]\,\mathrm{d}x - m_1(2v_{1E}\dot{v}_{1E}\dot{\theta}_1 + v_{1E}^2\ddot{\theta}_1$$
$$+ l_1\ddot{v}_{1E} + l_1^2\ddot{\theta}_1) - m_2(2v_{1E}\dot{v}_{1E}\dot{\theta}_1 + v_{1E}^2\ddot{\theta}_1 + l_1\ddot{v}_{1E} + l_1^2\ddot{\theta}_1 + 2v_{2E}\dot{v}_{2E}\dot{\theta}_1$$
$$+ v_{2E}^2\ddot{\theta}_1 + 2v_{2E}\dot{v}_{2E}\dot{v}_{1E}' + v_{2E}^2\ddot{v}_{1E}' + 2v_{2E}\dot{v}_{2E}\dot{\theta}_2 + v_{2E}^2\ddot{\theta}_2 + l_2^2\ddot{v}_{2E}$$
$$+ l_2^2\ddot{\theta}_1 + l_2^2\ddot{v}_{1E}' + l_2^2\ddot{\theta}_2) - J_1\ddot{\theta}_1 - J_2(\ddot{\theta}_1 + \ddot{v}_{1E}' + \ddot{\theta}_2) + \tau_1$$
$$- \lambda\{(b + c\mu \operatorname{sign}(\lambda))[l_1 \sin\theta_1 + v_{1E}\cos\theta_1 + l_2\sin(\theta_1 + v_{2E}' + \theta_2)$$
$$+ v_{2E}\cos(\theta_1 + v_{2E}' + \theta_2)] - (c - b\mu \operatorname{sign}(\lambda))[l_1\cos\theta_1 - v_{1E}\sin\theta_1$$
$$+ l_2\cos(\theta_1 + v_{2E}' + \theta_2) - v_{2E}\sin(\theta_1 + v_{2E}' + \theta_2)]\} = 0. \tag{51}$$

The governing equation of the joint angle θ_2 is

$$\int_0^{l_2} [\rho_2A_2(-x^2\ddot{\theta}_1 - x^2\ddot{\theta}_2 - 2v_2\dot{v}_2\dot{\theta}_1 - v_2^2\ddot{\theta}_1 - 2v_2\dot{v}_2\dot{v}_{1E}' - v_2^2\ddot{v}_{1E}' - 2v_2\dot{v}_2\dot{\theta}_2$$
$$- v_2^2\ddot{\theta}_2 - x\ddot{v}_2 - x^2\ddot{v}_{1E}')]\,\mathrm{d}x - m_2(2v_{2E}\dot{v}_{2E}\dot{\theta}_1 + v_{2E}^2\ddot{\theta}_1 + 2v_{2E}\dot{v}_{2E}\dot{v}_{1E}' + v_{2E}^2\dot{v}_{1E}'$$
$$+ 2v_{2E}\dot{v}_{2E}\dot{\theta}_2 + v_{2E}^2\ddot{\theta}_2 + l_2\ddot{v}_{2E} + l_2^2\ddot{\theta}_1 + l_2^2\ddot{v}_{1E}' + l_2^2\ddot{\theta}_2) - J_2(\dot{\theta}_1 + \dot{\theta}_2 + \dot{v}_{1E}') + \tau_2$$

$$- \lambda\{(b + c\mu \operatorname{sign}(\lambda))l_2 \sin(\theta_1 + v'_{2E} + \theta_2) - v_{2E} \cos(\theta_1 + v'_{2E} + \theta_2)] + (c - b\mu \operatorname{sign}(\lambda))[(l_2 + u_{2E}) \cos(\theta_1 + v'_{2E} + \theta_2) - v_{2E} \times \sin(\theta_1 + v'_{2E} + \theta_2)]\} = 0. \tag{52}$$

If the flexibility of the link 1, geometric constraint and tip mass are neglected, the above equations (48), (51) and (52), coincide with those derived by Low and Vidyasagar [3].

2.4. RIGID BODY MODEL

When all flexible deformations are neglected, one obtains the rigid body motion of the two-link manipulators. By assuming the two links have uniform cross-sectional areas, and integrating equations (51) and (52), one obtains the governing equations for θ_1 and θ_2, respectively, as

$$\tfrac{1}{3}(-l_1^2 m_3 \ddot{\theta}_1 - l_2^2 m_4(\ddot{\theta}_1 + \ddot{\theta}_2)) - m_1 l_1^2 \ddot{\theta}_1 - m_2(l_1^2 \ddot{\theta} + l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)) + \tau_1 - \lambda\{(b + c\mu \operatorname{sign}(\lambda))[l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)] - (c - b\mu \operatorname{sign}(\lambda))[l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)] = 0, \tag{53}$$

$$-l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)(\tfrac{1}{3}m_4 + m_2) + \tau_2 - \lambda[(b + c\mu \operatorname{sign}(\lambda))l_2 \sin(\theta_1 + \theta_2) + (c - b\mu \operatorname{sign}(\lambda))l_2 \cos(\theta_1 + \theta_2)] = 0, \tag{54}$$

where $m_3 = \rho_1 A_1 l_1$ and $m_4 = \rho_2 A_2 l_2$ are the masses of the uniform links 1 and 2 respectively.

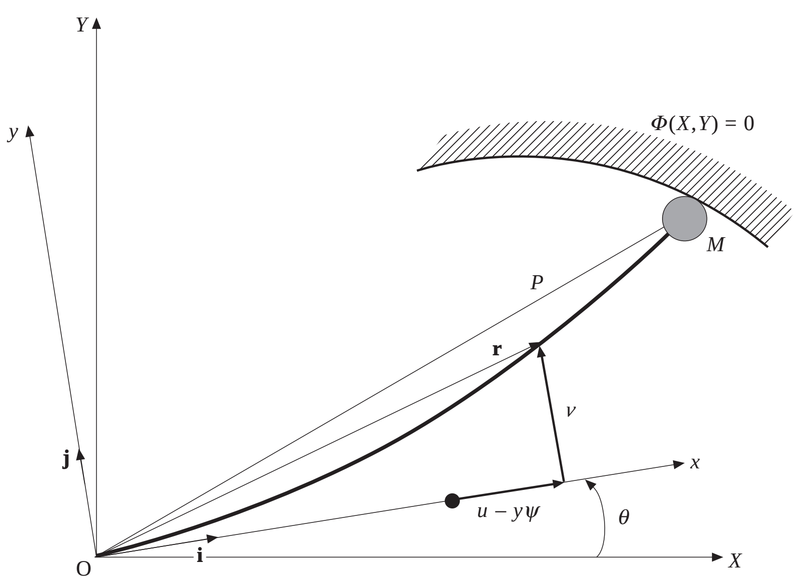

Figure 2. Model of a single-link flexible manipulator.

3. SINGLE-LINK FLEXIBLE MANIPULATOR

By considering a single-link manipulator in contact with the constrained surface, $l_2 = l$, $m_2 = M$, $\rho_2 = \rho$, $A_2 = A$, $E_2 = E$, $K_2 = K$, $G_2 = G$, $I_2 = I$, $u_2 = u$, $v_2 = v$, $\psi_2 = \psi$, $\tau_2 = \tau$, $J_2 = J$ and $\theta_2 = \theta$ and the effects of link 1 can be neglected. The physical model of a single-link flexible manipulator is illustrated in Figure 2, where (X, Y) is the base co-ordinate system, and $[\mathbf{i}, \mathbf{j}]$ are the orthogonal unit vectors of rotating co-ordinate with origin O. Using Timoshenko beam theory, Euler beam theory, simple–flexure model and rigid-body model, the above governing equations and boundary conditions can be simplified as follows.

3.1. TIMOSHENKO BEAM THEORY

By using Timoshenko beam theory, the governing equations of θ, u, v and ψ, and the boundary conditions can be expressed as

$$\theta\colon \int_0^l [\rho A(\ddot{u}v - 2x\dot{u}\dot{\theta} - 2xu\ddot{\theta} - x^2\ddot{\theta} - 2u\dot{u}\dot{\theta} - u^2\ddot{\theta} - 2v\dot{v}\dot{\theta} - v^2\ddot{\theta} - x\ddot{v} - u\ddot{v})$$

$$- \rho I(2\psi\dot{\psi}\dot{\theta} + \psi^2\ddot{\theta} + \ddot{\psi} + \ddot{\theta})]\,\mathrm{d}x - M(-\ddot{u}_E v_E + 2v_E\dot{v}_E\dot{\theta} + v_E^2\ddot{\theta} + l\ddot{v}_E + u_E\ddot{v}_E + l^2\ddot{\theta}$$

$$+ 2l\dot{u}_E\dot{\theta} + 2lu_E\ddot{\theta} + 2u_E\dot{u}_E\dot{\theta} + u_E^2\ddot{\theta} - J\ddot{\theta} + \tau - \lambda\{(b + c\mu\,\mathrm{sign}\,(\lambda))[(l + u_E)\sin\theta$$

$$- v_E\cos\theta] + (c - b\mu\,\mathrm{sign}\,(\lambda))[(l + u_E)\cos\theta - v_E\sin\theta]\} = 0, \tag{55}$$

$$u\colon \rho A(x\dot{\theta}^2 + u\dot{\theta}^2 - \ddot{u} + v\ddot{\theta} + 2\dot{v}\dot{\theta}) + EA(u'' + v'v'') = 0, \tag{56}$$

$$v\colon \rho A(v\dot{\theta}^2 - \ddot{v} - x\ddot{\theta} - u\ddot{\theta} - 2\dot{u}\dot{\theta}) + EA[(u'' + v'v'')v' + (u' + \tfrac{1}{2}v'^2)v'']$$

$$KGA(v'' - \psi') = 0, \tag{57}$$

$$\psi\colon \rho I(\psi\dot{\theta}^2 - \ddot{\psi} - \ddot{\theta}) + KGA(v' - \psi) + EI\psi'' = 0, \tag{58}$$

$$u(0, t) = 0, \qquad v(0, t) = 0, \qquad \psi'(0, t) = 0, \qquad \psi'(l, t) = 0, \tag{59a–d}$$

$$M(l\dot{\theta}^2 + u_E\dot{\theta}^2 - \ddot{u}_E + 2\dot{v}_E\dot{\theta} + v_E\ddot{\theta}) - EA(u_E' + \tfrac{1}{2}v_E'^2)$$

$$+\lambda[(b + c\mu\,\mathrm{sign}\,(\lambda))\cos\theta + (c - b\mu\,\mathrm{sign}\,(\lambda))\sin\theta = 0, \tag{60}$$

$$M(-2\dot{u}_E\dot{\theta} + v_E\dot{\theta}^2 - \ddot{v}_E - l\ddot{\theta} - u_E\ddot{\theta}) - EA(u_E' + \tfrac{1}{2}v_E'^2)v_E'$$

$$-KGA(v_E' - \psi_E) - \lambda[(b + c\mu\,\mathrm{sign}\,(\lambda))\sin\theta - (c - b\mu\,\mathrm{sign}\,(\lambda))\cos\theta)] = 0. \tag{61}$$

The above equations are similar to those derived by Wang and Guan [5] in which the geometric constraint has not been considered.

3.2. EULER BEAM THEORY

By using Euler beam theory, the governing equations of θ, u and v, and the boundary conditions can be expressed as

$$\theta : \int_0^1 \rho A(\ddot{u}v - 2x\dot{u}\dot{\theta} - 2xu\ddot{\theta} - x^2\ddot{\theta} - 2u\dot{u}\dot{\theta} - u^2\ddot{\theta} - 2v\dot{v}\dot{\theta} - v^2\ddot{\theta} - x\ddot{v} - u\ddot{v})\,\mathrm{d}x$$
$$- M(-\ddot{u}_E v_E + 2v_E\dot{v}_E\dot{\theta} + v_E^2\ddot{\theta} + l\ddot{v}_E + u_E\ddot{v}_E + l^2\ddot{\theta} + 2l\dot{u}_E\dot{\theta}$$
$$+ 2lu_E\ddot{\theta} + 2u_E\dot{u}_E\dot{\theta} + u_E^2\ddot{\theta} - J\ddot{\theta} + \tau - \lambda\{(b + c\mu \operatorname{sign}(\lambda))[(l + u_E)\sin\theta$$
$$- v_E\cos\theta] + (c - b\mu \operatorname{sign}(\lambda))[(l + u_E)\cos\theta - v_E\sin\theta]\} = 0, \qquad (62)$$

u: equation (56),

$$v: \rho A(v\dot{\theta}^2 - \ddot{v} - x\ddot{\theta} - u\ddot{\theta} - 2\dot{u}\dot{\theta}) + EA[(u'' + v'v'')v' + (u' + \tfrac{1}{2}v'^2)v''] - EIv'''' = 0, \qquad (63)$$

$$u(0, t) = 0, \qquad v(0, t) = 0, \qquad v''(0, t) = 0, \qquad v''(l, 0) = 0, \qquad (64\text{a–d})$$

$$M(v_E\dot{\theta}^2 - \ddot{v}_E - l\ddot{\theta} - 2\dot{u}_E\dot{\theta} - u_E\ddot{\theta}) - EA(u'_E + \tfrac{1}{2}v_E'^2)v'_E$$
$$+ EIv_E'''' - \lambda[(b + c\mu \operatorname{sign}(\lambda))\sin\theta - (c - b\mu \operatorname{sign}(\lambda))\cos\theta] = 0, \qquad (65)$$

and equation (60).

Without the constrained forces and the tip mass, the above equations (62–64) coincide with those derived by Yuan [2]. If setting $u = 0$ in the equations (62–65), the same differential equations were derived by Fung and Shi [15].

3.3. SIMPLE FLEXURAL MODEL

By using the simple-flexure model, the governing equations of θ and v, and the boundary conditions can be expressed as

$$\theta: \int_0^l \rho A(-v^2\ddot{\theta} - 2v\dot{v}\dot{\theta} - x^2\ddot{\theta} - x\ddot{v})\,\mathrm{d}x + M[-v_E^2\ddot{\theta} - 2v_E\dot{v}_E\theta - l\ddot{v}_E - l^2\ddot{\theta}]$$
$$+ \tau - J\ddot{\theta} - \lambda[(b + c\mu \operatorname{sign}(\lambda))(l\sin\theta + v_E\cos\theta)$$
$$+ (c - b\mu \operatorname{sign}(\lambda))((l + u(l, t))\cos\theta - v(l, t)\sin\theta)] = 0, \qquad (66)$$

$$v: \rho A(v\dot{\theta}^2 - \ddot{v} - x\ddot{\theta}) - [pv']' - EIv'''' = 0, \qquad (67)$$

$$v(0, t) = 0, \qquad v''(0, t) = 0, \qquad v''(l, t) = 0, \qquad (68\text{a–c})$$

$$M[v_E\dot{\theta}^2 - \ddot{v}_E - l\ddot{\theta}] - p(l, t) + EIv_E''' - \lambda[(c - b\mu \operatorname{sign}(\lambda))\sin\theta$$
$$- (b + c\mu \operatorname{sign}(\lambda))\cos\theta] = 0, \qquad (69)$$

where

$$p(x, t) = p(l, t) - \int_x^l \frac{\partial}{\partial x} p(x, t)\,\mathrm{d}x$$

$$= M(\dot{v}_E\dot{\theta} + \dot{v}_E\dot{\theta} + v_E\ddot{\theta} + l\dot{\theta}^2) + \lambda[(b + c\mu \operatorname{sign}(\lambda)) \cos\theta + (c - b\mu \operatorname{sign}(\lambda)) \sin\theta] + [\rho A(\dot{v}\dot{\theta} + x\dot{\theta}^2 + v\ddot{\theta} + \dot{v}\dot{\theta})]_x^l. \qquad (70)$$

If one neglects the terms of the constrained force and the tip mass and sets $u = 0$ in the Euler–Bernoulli beam equations (62–64), the equations for the simple flexible model coincide with those derived in references [6, 7]. Thus, one can confirm that the results of the present derivation are correct.

3.4. RIGID BODY MODEL

All the flexible deformations are not considered in the rigid body model. The governing equation of rotation for the rigid manipulator with constrained force is

$$-l^2\ddot{\theta}(\tfrac{1}{3}M^* + M) + \tau - \lambda[(b + c\mu \operatorname{sign}(\lambda))l \sin\theta + (c - b\mu \operatorname{sign}(\lambda))l \cos\theta] = 0, \qquad (71)$$

where $M^* = \rho Al$ is the mass of the uniform rigid link.

4. DISCUSSION

The main objective of this paper is to derive the dynamic equations of the constrained flexible manipulator with a tip mass by the use of various beam theories. The reduction process was shown by starting with the Timoshenko beam theory and going through the Euler beam theory, simple flexible beam model and, finally, the rigid body beam.

From the above governing equations and boundary conditions, several important observations can be made:

(1) In the simple-flexible model, the v governing equation is also non-linear.

(2) The resultant equations of the two-link manipulators are non-linear and include Coriolis and centrifugal effects.

(3) The rigid body motion and flexible vibration are non-linearly coupled in all flexible beam models. Even though the geometric non-linearity is absent, these are also coupled. Thus, a complete analysis of the flexible manipulators should include both the rigid body motions and flexible vibrations.

(4) Due to the rigid body motion of the rotor and the flexible vibrations of the manipulator being coupled, it provides the opportunity that the flexible vibrations of the manipulator can be suppressed by controlling the input torque developed by the motor.

(5) The mass of tip load is the significant factor for the effect of tip load on vibration. When a load with a large mass is grasped by a flexible manipulator, the dynamics of the manipulator will be changed dramatically.

(6) The contact force is composed of the generalized normal and friction forces. The generalized normal force is the gradient of the constraint surface multiplied with the Lagrange multiplier. According to Coulomb's law, the friction force is represented as the product of the magnitude of the normal force and a friction coefficient. It is depicted in equations (9) and (10) that the friction force is perpendicular to the normal force.

(7) It is seen in equation (8) that the constrained condition Φ includes the variables of rotation and elastic deformations. The reaction forces come from not only elastic deformations but also rotation of the rigid-body motion.

(8) It is well known that the generalized normal force is normal to the constraint surface so that no work is done. When Hamilton's principle is employed to derive the equations of motion, the variations of kinetic energy and potential energy and the virtual work done by external forces and the constraint forces are used. However, the zero virtual work done by the normal force is still considered in Hamilton's principle to investigate the effect of the normal force on the system.

(9) The meaning of the Lagrange multiplier λ is a scalar, which represents the connection of the constraint force and kinematics constraint. The product of the Lagrange multiplier with the magnitude of the gradient of the constraint equals in magnitude the normal force imposing the constraint. The sign of the Lagrange multiplier decides if the generalized normal force is directed along the positive or negative normal to the constrained surface.

(10) The terms with the Lagrange multiplier λ include the normal and friction forces, which do not appear in the governing equations of the Timoshenko and Euler beam theories, but appear in the governing equation and boundary condition of the simple flexure model and the rigid body motions of all four models.

5. CONCLUSIONS

On the basis of the four dynamic models for the single-link and two-link flexible manipulators, a comprehensive derivation of the geometric constraint and the tip mass on the dynamic formulation of the manipulators have been conducted completely in this paper. The formulation is based on the expressions of the kinetic and strain energies and the virtual works done by the external and constrained forces. In dealing with the constrained problem, a procedure has been presented for incorporating the generalized normal and friction forces into the models.

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